

The χ -part of the analytic class number formula, for global function fields.

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Abstract

Let F/k be a finite abelian extension of global function fields, totally split at a distinguished place ∞ of k . We show that a complex Gras conjecture holds for Stark units, and we derive a refined analytic class number formula.

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1 Introduction.

Let k be a global function field with constant field \mathbb{F}_q , and let ∞ be a distinguished place of k . We write k_∞ for the completion of k at ∞ . For any finite abelian extension K/k , let \mathcal{O}_K be the ring of functions of k which are regular outside the places of K sitting above ∞ . We denote by $\text{Cl}(\mathcal{O}_K)$ the ideal-class group of \mathcal{O}_K . If $K \subseteq k_\infty$ we define in section 2 a group \mathcal{E}_K of Stark units, which have finite index in \mathcal{O}_K^\times , the group of units of \mathcal{O}_K . This group \mathcal{E}_K has already been studied in [3], [4].

We fix a finite abelian extension $F \subseteq k_\infty$ of k , with Galois group G , and degree g . In [9], for every nontrivial irreducible rational character ψ of G , we established that

$$\#(\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} (\mathcal{O}_F^\times / \mathcal{E}_F))_\psi = \#(\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_\psi, \quad (1.1)$$

where $\mathbb{Z}_{\langle g \rangle} := \mathbb{Z}[g^{-1}]$, and where the index ψ means we take the ψ -parts. In [5] we used Euler systems to prove that \mathcal{E}_F satisfies the Gras conjecture,

$$\#(\mathbb{Z}_p \otimes_{\mathbb{Z}} (\mathcal{O}_F^\times / \mathcal{E}_F))_\psi = \#(\mathbb{Z}_p \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_\psi,$$

for all prime number $p \nmid qg$ and all irreducible nontrivial \mathbb{Q}_p -character ψ (but we were not able to prove the conjecture in the special case where the following conditions are simultaneously satisfied: $p \mid \#\text{Cl}(\mathcal{O}_k)$, ψ is a conjugate of the Teichmüller character, $\mu_p \not\subset k$ and $\mu_p \subset F$, where μ_p is the group of p -th roots of unity in the separable closure of k).

Let μ_g be the group of g -th roots of unity in the field of complex numbers. Let \mathcal{O} be the integral closure of $\mathbb{Z}_{\langle g \rangle}$ in $\mathbb{Q}(\mu_g)$. For any module M over a commutative ring

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A , we denote by $\text{Fit}_A(M)$ the Fitting ideal of M . Using the properties of Rubin-Stark units stated in [7] and [6], we prove here a complex version of the Gras conjecture for \mathcal{E}_F (Theorem 4.1). More precisely, we prove that for every nontrivial complex character χ of G , we have

$$\text{Fit}_{\mathcal{O}}(\mathcal{O} \otimes_{\mathbb{Z}} (\mathcal{O}_F^{\times}/\mathcal{E}_F))_{\chi} = \text{Fit}_{\mathcal{O}}(\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_{\chi}, \quad (1.2)$$

which is a refinement of (1.1). From (1.2), one can easily deduce that the classical Gras conjecture holds for all prime number $p \nmid g$. Thus Theorem 4.1 expands the result obtained in [5], in particular it shows that the conjecture holds for p equal to the characteristic of k , and also for the conjugates of the Teichmüller character in the above special case. Let us also mention the analogy between this theorem and a recent result of P. Buckingham (see [1, Theorem 7.1]), who is concerned with Rubin-Stark elements in cyclic extensions of totally real number fields.

Also we can combine this complex Gras conjecture (1.2) with the computations made in [9]. Thus we derive a « χ -part» version of the analytic class number formula (Theorem 4.2), connecting the Fitting ideal of the χ -part of the \mathcal{O} -module $\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F)$ to the value at 0 of an L -function attached to $\overline{\chi}$, for any non trivial irreducible complex character χ of G .

2 Stark units in function fields.

Let d be the degree of ∞ over \mathbb{F}_q . If \mathfrak{m} is a nonzero ideal of \mathcal{O}_k , then we denote by $H_{\mathfrak{m}} \subseteq k_{\infty}$ the maximal abelian extension of k contained in k_{∞} , such that the conductor of $H_{\mathfrak{m}}/k$ divides \mathfrak{m} . The function field version of the abelian conjectures of Stark, proved by P. Deligne in [8] by using étale cohomology or by D. Hayes in [2] by using Drinfel'd modules, claims that, for any proper nonzero ideal \mathfrak{m} of \mathcal{O}_k , there exists an element $\varepsilon_{\mathfrak{m}} \in H_{\mathfrak{m}}$, unique up to roots of unity, such that

i) The extension $H_{\mathfrak{m}}(\varepsilon_{\mathfrak{m}}^{1/w_{\infty}})/k$ is abelian, where $w_{\infty} := q^d - 1$. Moreover, it is unramified outside $S_{\mathfrak{m}}$, where $S_{\mathfrak{m}}$ is the set containing ∞ and the places of k which divide \mathfrak{m} .

ii) If \mathfrak{m} is divisible by two prime ideals then $\varepsilon_{\mathfrak{m}}$ is a unit of $\mathcal{O}_{H_{\mathfrak{m}}}$. If $\mathfrak{m} = \mathfrak{q}^e$, where \mathfrak{q} is a prime ideal of \mathcal{O}_k and e is a positive integer, then

$$\varepsilon_{\mathfrak{m}} \mathcal{O}_{H_{\mathfrak{m}}} = (\mathfrak{q})_{\mathfrak{m}}^{\frac{w_{\infty}}{w_k}},$$

where $w_k := q - 1$ and $(\mathfrak{q})_{\mathfrak{m}}$ is the product of the prime ideals of $\mathcal{O}_{H_{\mathfrak{m}}}$ which divide \mathfrak{q} .

iii) We have

$$L_{\mathfrak{m}}(0, \chi) = \frac{1}{w_{\infty}} \sum_{\sigma \in \text{Gal}(H_{\mathfrak{m}}/k)} \chi(\sigma) v_{\infty}(\varepsilon_{\mathfrak{m}}^{\sigma}) \quad (2.1)$$

for all complex irreducible characters of $\text{Gal}(H_{\mathfrak{m}}/k)$, where v_{∞} is the normalized valuation of k_{∞} .

Let us recall that $s \mapsto L_{\mathfrak{m}}(s, \chi)$ is the L -function associated to χ , defined for the complex numbers s such that $\text{Re}(s) > 1$ by the Euler product

$$L_{\mathfrak{m}}(s, \chi) = \prod_{v \nmid \mathfrak{m}} (1 - \chi(\sigma_v) N(v)^{-s})^{-1},$$

where v describes the set of places of k not dividing \mathfrak{m} . For such a place, σ_v and $N(v)$ are the Frobenius automorphism of $H_{\mathfrak{m}}/k$ and the order of the residue field at v respectively. Let us remark that $\sigma_{\infty} = 1$ and $N(\infty) = q^d$.

For any finite abelian extension L of k we denote by $\mathcal{J}_L \subseteq \mathbb{Z}[\text{Gal}(L/k)]$ the annihilator of $\mu(L)$, the group of roots of unity in L . The description of \mathcal{J}_L given in [2, Lemma 2.5] and the property $i)$ of $\varepsilon_{\mathfrak{m}}$ implies that for any $\eta \in \mathcal{J}_{H_{\mathfrak{m}}}$ there exists $\varepsilon_{\mathfrak{m}}(\eta) \in H_{\mathfrak{m}}$ such that

$$\varepsilon_{\mathfrak{m}}(\eta)^{w_{\infty}} = \varepsilon_{\mathfrak{m}}^{\eta}. \quad (2.2)$$

Definition 2.1 Let \mathcal{P}_F be the subgroup of F^{\times} generated by $\mu(F)$ and by all the norms

$$\varepsilon_{F,\mathfrak{m}}(\eta) := N_{H_{\mathfrak{m}}/H_{\mathfrak{m}} \cap F}(\varepsilon_{\mathfrak{m}}(\eta)),$$

where \mathfrak{m} is any nonzero proper ideal of \mathcal{O}_k , and $\eta \in \mathcal{J}_{H_{\mathfrak{m}}}$. Then we set

$$\mathcal{E}_F := \mathcal{P}_F \cap \mathcal{O}_F^{\times}.$$

3 Preliminary lemmas.

For any finite group H , let us denote by \widehat{H} the group of complex irreducible characters of H . Then for every $\chi \in \widehat{H}$ we set $e_{\chi} := \frac{1}{\#H} \sum_{\sigma \in G} \chi(\sigma) \sigma^{-1}$. In case $H = G$, then e_{χ} belongs to $\mathcal{O}[G]$. Moreover, if $\zeta \in \mu_g$ is such that $\zeta \neq 1$, then $(1 - \zeta) \in \mathcal{O}^{\times}$, thanks to the formula $g = \prod_{\substack{\zeta \in \mu_g \\ \zeta \neq 1}} (1 - \zeta)$. For any $\mathcal{O}[G]$ -module M let $M_{\chi} := e_{\chi} M$ be the χ -part of M .

For K/k a finite abelian extension, and $S \neq \emptyset$ a finite set of places of k , let $\mathcal{O}_{K,S}$ be the Dedekind ring of the S -integers of K , i.e the functions $f \in K$ which only poles are at the places sitting over S . Let $\text{Cl}(\mathcal{O}_{K,S})$ be the ideal class group of $\mathcal{O}_{K,S}$.

Lemma 3.1 Let \mathfrak{m} be a nonzero ideal of \mathcal{O}_k , and let $\chi \in \widehat{G}$, $\chi \neq 1$. Assume that for every prime ideal \mathfrak{p} of \mathcal{O}_k which divides \mathfrak{m} , χ is not trivial on the decomposition group $D_{\mathfrak{p}}$ of \mathfrak{p} in F/k . Let $S_{\mathfrak{m}}$ be the set of places of k which contains ∞ and all the prime divisors of \mathfrak{m} . Then in the category of $\mathcal{O}[G]$ -modules, we have

$$(\mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}_{F,S_{\mathfrak{m}}}^{\times})_{\chi} = (\mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}_F^{\times})_{\chi} \quad \text{and} \quad (\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_{F,S_{\mathfrak{m}}}))_{\chi} \simeq (\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_{\chi}.$$

Proof. Let \mathcal{S} be the $\mathbb{Z}[G]$ -module generated by the prime ideals of \mathcal{O}_F dividing $\mathfrak{m}\mathcal{O}_F$. Let $\bar{\mathcal{S}}$ be the image of \mathcal{S} in $\text{Cl}(\mathcal{O}_F)$. We have the following exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bar{\mathcal{S}} & \longrightarrow & \text{Cl}(\mathcal{O}_F) & \longrightarrow & \text{Cl}(\mathcal{O}_{F,S_{\mathfrak{m}}}) \longrightarrow 0, \\ 0 & \longrightarrow & \mathcal{O}_F^{\times} & \longrightarrow & \mathcal{O}_{F,S_{\mathfrak{m}}}^{\times} & \longrightarrow & \mathcal{S}, \\ & & & & x & \longmapsto & \prod_{\mathfrak{p} \in \mathcal{S}} \mathfrak{p}^{v_{\mathfrak{p}}(x)}, \end{array}$$

where for all $\mathfrak{p} \in \mathcal{S}$, $v_{\mathfrak{p}}$ is the normalized valuation at \mathfrak{p} . Since \mathcal{O} is \mathbb{Z} -flat, all we have to show is $(\mathcal{O} \otimes_{\mathbb{Z}} \mathcal{S})_{\chi} = 0$. Let $\mathfrak{p} \in \mathcal{S}$. There is $\gamma \in G$ such that $\mathfrak{p}^{\gamma} = \mathfrak{p}$ and $\chi(\gamma) \neq 1$. Then $(\chi(\gamma) - 1)e_{\chi}(1 \otimes \mathfrak{p}) = e_{\chi}(1 \otimes \mathfrak{p}^{\gamma-1}) = 0$, with $(\chi(\gamma) - 1) \in \mathcal{O}^{\times}$, hence $e_{\chi}(1 \otimes \mathfrak{p}) = 0$. \square

For $K \subseteq k_{\infty}$ a finite abelian extension of k , let $\ell_K : K^{\times} \rightarrow \mathbb{Z}[\text{Gal}(K/k)]$ be the $\text{Gal}(K/k)$ -equivariant map defined by

$$\ell_K(x) := \sum_{\sigma \in \text{Gal}(K/k)} v_{\infty}(x^{\sigma}) \sigma^{-1}.$$

Lemma 3.2 *Let \mathfrak{m} be a nonzero ideal of \mathcal{O}_k , and $\alpha \in \mathcal{J}_{H_m}$. We set $C_m := \text{Cl}(\mathcal{O}_{F \cap H_m, S_m})$. Then*

$$\ell_F(\varepsilon_{F, \mathfrak{m}}(\alpha)) \in \text{Fit}_{\mathbb{Z}_{\langle g \rangle}[G]}(C_m) \ell_F(\mathcal{O}_{F \cap H_m, S_m}^\times).$$

Proof. By the description of \mathcal{J}_{H_m} given in [2], Lemma 2.5, we know that there are a finite set T of nonzero prime ideals of \mathcal{O}_k , and a family $(\alpha_{\mathfrak{p}})_{\mathfrak{p} \in T} \in \mathbb{Z}[\text{Gal}(H_m/k)]^T$, such that $S_m \cap T = \emptyset$ and $\alpha = \sum_{\mathfrak{p} \in T} \alpha_{\mathfrak{p}} (1 - N(\mathfrak{p})\sigma_{\mathfrak{p}}^{-1})$, where $\sigma_{\mathfrak{p}}$ is the Frobenius of \mathfrak{p} in H_m/k . It suffices to show $\ell_F(\varepsilon_{F, \mathfrak{m}}(1 - N(\mathfrak{p})\sigma_{\mathfrak{p}}^{-1})) \in \text{Fit}_{\mathbb{Z}_{\langle g \rangle}[G]}(C_m) \ell_F(\mathcal{O}_{F \cap H_m, S_m}^\times)$, for a fixed nonzero prime ideal \mathfrak{p} of \mathcal{O}_k , with $\mathfrak{p} \notin S_m$.

For any abelian extension K of k , we denote by $U_{K, \mathfrak{m}, \mathfrak{p}}$ the group of units of \mathcal{O}_{K, S_m} which are congruent to 1, modulo all primes above \mathfrak{p} . From (2.2) we deduce that $\varepsilon_m(1 - N(\mathfrak{p})\sigma_{\mathfrak{p}}^{-1})$ can be chosen such that $\varepsilon_m(1 - N(\mathfrak{p})\sigma_{\mathfrak{p}}^{-1}) \in U_{H_m, \mathfrak{m}, \mathfrak{p}}$. For $\chi \in \widehat{\text{Gal}(H_m/k)}$, we define the meromorphic function $s \mapsto L_{\mathfrak{m}, \mathfrak{p}}(s, \bar{\chi})$ by

$$L_{\mathfrak{m}, \mathfrak{p}}(s, \bar{\chi}) := (1 - N(\infty)^{-s}) L_m(s, \bar{\chi}) (1 - N(\mathfrak{p})^{1-s} \bar{\chi}(\sigma_{\mathfrak{p}})).$$

Derivating, and using the property *iii*) of Stark units, we obtain

$$\begin{aligned} L'_{\mathfrak{m}, \mathfrak{p}}(0, \bar{\chi}) &= d.\ln(q) L_m(0, \bar{\chi}) (1 - N(\mathfrak{p})\chi(\sigma_{\mathfrak{p}}^{-1})) \\ &= d.\ln(q) \chi(\ell_{H_m}(\varepsilon_m(1 - N(\mathfrak{p})\sigma_{\mathfrak{p}}^{-1}))), \end{aligned} \quad (3.1)$$

where χ is extended to $\mathbb{C}[\text{Gal}(H_m/k)]$ by linearity. For $s \in \mathbb{C}$, we set

$$\Theta_{\mathfrak{m}, \mathfrak{p}}(s) = \sum_{\chi \in \widehat{\text{Gal}(H_m/k)}} L_{\mathfrak{m}, \mathfrak{p}}(s, \bar{\chi}) e_{\chi} \quad \text{in } \mathbb{C}[\text{Gal}(H_m/k)],$$

wherever it is defined. From (3.1), we have

$$\begin{aligned} \Theta'_{\mathfrak{m}, \mathfrak{p}}(0) &= d.\ln(q) \ell_{H_m}(\varepsilon_m(1 - N(\mathfrak{p})\sigma_{\mathfrak{p}}^{-1})) \\ &= - \sum_{\gamma \in \text{Gal}(H_m/k)} \ln \left(\left| \varepsilon_m(1 - N(\mathfrak{p})\sigma_{\mathfrak{p}}^{-1})^{\gamma^{-1}} \right|_{\infty} \right) \gamma. \end{aligned}$$

We have just verified that $\varepsilon_m(1 - N(\mathfrak{p})\sigma_{\mathfrak{p}}^{-1})$ satisfies [6, Theorem 0], for $(H_m, S_m, \{\mathfrak{p}\}, 1)$. Then $\varepsilon_{F, \mathfrak{m}}(1 - N(\mathfrak{p})\sigma_{\mathfrak{p}}^{-1})$ satisfies [6, Theorem 0], for $(H_m \cap F, S_m, \{\mathfrak{p}\}, 1)$, and we have

$$\ell_F(\varepsilon_{F, \mathfrak{m}}(1 - N(\mathfrak{p})\sigma_{\mathfrak{p}}^{-1})) \in \text{Fit}_{\mathbb{Z}_{\langle g \rangle}[G]}(\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_{F \cap H_m, S_m}, \mathfrak{p})) \ell_F(U_{F \cap H_m, \mathfrak{m}, \mathfrak{p}}^\times),$$

where $\text{Cl}(\mathcal{O}_{F \cap H_m, S_m}, \mathfrak{p})$ is the quotient of fractional ideals of $\mathcal{O}_{F \cap H_m}$ by those principal fractional ideals, which are generated by an element congruent to 1 modulo \mathfrak{p} . But $\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} C_m$ is a quotient of $\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_{F \cap H_m, S_m}, \mathfrak{p})$, and $\mathbb{Z}_{\langle g \rangle}[G]$ is a finite product of Dedekind rings, so

$$\text{Fit}_{\mathbb{Z}_{\langle g \rangle}[G]}(\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_{F \cap H_m, S_m}, \mathfrak{p})) \subseteq \text{Fit}_{\mathbb{Z}_{\langle g \rangle}[G]}(\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} C_m),$$

and we are done. \square

Definition 3.1 *Let Ω be the $\mathbb{Z}[G]$ -submodule of F^\times generated by $\mu(F)$ and by the elements $\varepsilon_{F, \mathfrak{m}} := N_{H_m/F \cap H_m}(\varepsilon_m)$, where \mathfrak{m} is any nonzero proper ideal of \mathcal{O}_k .*

For any $\chi \in \widehat{G}$, let F_χ be the subfield of F fixed by $\text{Ker}(\chi)$, and \mathfrak{f}_χ be the conductor of F_χ . χ_{pr} denotes the character of $\text{Gal}(H_{\mathfrak{f}_\chi}/k)$ defined by χ . Assume that χ is nontrivial. Then by [9, Proposition 3.1], we have

$$(\mathcal{O}\ell_F(\Omega))_\chi = \mathcal{O}w_\infty L_{\mathfrak{f}_\chi}(0, \bar{\chi}_{\text{pr}}) e_\chi. \quad (3.2)$$

Moreover one can easily relate $\ell_F(\Omega)$ to $\ell_F(\mathcal{E}_F)$ (see [9, (3.13)]). If $\chi \neq 1$, we have

$$(\mathcal{O}\ell_F(\mathcal{E}_F))_\chi = (\mathcal{O}w_\infty^{-1} \mathcal{J}_F \ell_F(\Omega))_\chi. \quad (3.3)$$

Lemma 3.3 *Let $\chi \in \widehat{G}$, $\chi \neq 1$. There is a nonzero ideal \mathfrak{m} of \mathcal{O}_k , satisfying the following properties:*

- i) \mathfrak{m} is divisible by at least two distinct prime ideals,*
- ii) $F_\chi \subseteq H_{\mathfrak{m}}$,*
- iii) for each prime ideal \mathfrak{p} which divides \mathfrak{m} , χ is not trivial on the decomposition group $D_{\mathfrak{p}}$ of \mathfrak{p} in F/k .*
- iv) As an \mathcal{O} -module, $(\mathcal{O} \otimes_{\mathbb{Z}} \mathcal{E}_F)_\chi$ is generated by $(\mathcal{O} \otimes_{\mathbb{Z}} \mu(F))_\chi$ and by the elements $e_\chi(1 \otimes \varepsilon_{F,\mathfrak{m}}(\alpha))$, where $\alpha \in \mathcal{J}_{H_{\mathfrak{m}}}$.*

Proof. Since $\chi \neq 1$, we can find two distinct nonzero prime ideals \mathfrak{p} and \mathfrak{q} of \mathcal{O}_k , unramified in F/k , such that

$$\bar{\chi}_{\text{pr}}(\sigma_{\mathfrak{p}}) \neq 1 \quad \text{and} \quad \bar{\chi}_{\text{pr}}(\sigma_{\mathfrak{q}}) \neq 1.$$

We set $\mathfrak{m} = \mathfrak{f}_\chi \mathfrak{p} \mathfrak{q}$. Obviously, conditions *i)*, *ii)* and *iii)* are satisfied. As in the proof of [9, Proposition 3.1], from the property *iii)* of Stark units we obtain

$$\ell_F(\varepsilon_{F,\mathfrak{m}}) e_\chi = [F : F \cap H_{\mathfrak{m}}] w_\infty L_{\mathfrak{f}_\chi}(0, \bar{\chi}_{\text{pr}}) (1 - \bar{\chi}_{\text{pr}}(\sigma_{\mathfrak{p}})) (1 - \bar{\chi}_{\text{pr}}(\sigma_{\mathfrak{q}})) e_\chi.$$

Since $[F : F \cap H_{\mathfrak{m}}]$, $1 - \bar{\chi}_{\text{pr}}(\sigma_{\mathfrak{p}})$ and $1 - \bar{\chi}_{\text{pr}}(\sigma_{\mathfrak{q}})$ belong to \mathcal{O}^\times , and by (3.2), we have

$$\mathcal{O}\ell_F(\varepsilon_{F,\mathfrak{m}}) e_\chi = \mathcal{O}w_\infty L_{\mathfrak{f}_\chi}(0, \bar{\chi}_{\text{pr}}) e_\chi = \mathcal{O}\ell_F(\Omega) e_\chi. \quad (3.4)$$

From (3.4) and (3.3), we deduce

$$\mathcal{O}\mathcal{J}_F w_\infty^{-1} \ell_F(\varepsilon_{F,\mathfrak{m}}) e_\chi = \mathcal{O}\mathcal{J}_F w_\infty^{-1} \ell_F(\Omega) e_\chi = \mathcal{O}\ell_F(\mathcal{E}_F) e_\chi,$$

and condition *iv)* follows. □

To go further we need some preliminary remarks.

Remark 3.1 *For any $\mathcal{O}[G]$ -module M , we have $\text{Fit}_{\mathcal{O}[G]}(M) = \sum_{\chi \in \widehat{G}} \text{Fit}_{\mathcal{O}}(M_\chi) e_\chi$.*

Remark 3.2 *Let H be a sub-group of G . Let M and N be two G -modules, and $\psi : M \rightarrow N$ be a G -equivariant map. If $\text{Cok}(\Psi) := N/\text{Im}(\Psi)$ is annihilated by $\#(H)$ then we derive from Ψ a surjective map*

$$\Psi_{\mathcal{O}} : \mathcal{O} \otimes_{\mathbb{Z}} M \twoheadrightarrow \mathcal{O} \otimes_{\mathbb{Z}} N.$$

Let us assume, in addition, that $\text{Ker}(\Psi)$ is annihilated by $\Sigma\sigma$, $\sigma \in H$. Then, for every $\chi \in \widehat{G}$ trivial on H , the restriction of $\Psi_{\mathcal{O}}$ gives an isomorphism

$$(\mathcal{O} \otimes_{\mathbb{Z}} M)_\chi \simeq (\mathcal{O} \otimes_{\mathbb{Z}} N)_\chi.$$

As a particular case, for any subextension K/k of F/k and $H = \text{Gal}(F/K)$, and any nonzero ideal \mathfrak{m} of \mathcal{O}_k , the norm maps give isomorphisms

$$(\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_{F,S_{\mathfrak{m}}}))_{\chi} \xrightarrow{\sim} (\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_{K,S_{\mathfrak{m}}}))_{\chi} \quad \text{and} \quad (\mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}_{F,S_{\mathfrak{m}}}^{\times})_{\chi} \xrightarrow{\sim} (\mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}_{K,S_{\mathfrak{m}}}^{\times})_{\chi},$$

for any $\chi \in \widehat{G}$ which is trivial on H . Since $\#(H) \in \mathcal{O}^{\times}$ we deduce that for such a character, the canonical inclusion also gives an equality

$$(\mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}_{K,S_{\mathfrak{m}}}^{\times})_{\chi} = (\mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}_{F,S_{\mathfrak{m}}}^{\times})_{\chi}.$$

Proposition 3.1 *Let $\chi \in \widehat{G}$, $\chi \neq 1$. We have*

$$(\mathcal{O}\ell_F(\mathcal{E}_F))_{\chi} \subseteq \text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_{\chi} \right) (\mathcal{O}\ell_F(\mathcal{O}_F^{\times}))_{\chi}.$$

Proof. We choose an ideal \mathfrak{m} of \mathcal{O}_k satisfying the four conditions of Lemma 3.3. Because of Lemma 3.3, *iv*), it is sufficient to show that for $\alpha \in \mathcal{J}_{H_{\mathfrak{m}}}$, we have

$$\ell_F(\varepsilon_{F,\mathfrak{m}}(\alpha))e_{\chi} \in \text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_{\chi} \right) (\ell_F(\mathcal{O}_F^{\times}))_{\chi}.$$

By Lemma 3.2, we know that

$$\ell_F(\varepsilon_{F,\mathfrak{m}}(\alpha)) \in \text{Fit}_{\mathbb{Z}[G]}(C_{\mathfrak{m}}) \ell_F(\mathcal{O}_{F \cap H_{\mathfrak{m}}, S_{\mathfrak{m}}}^{\times}).$$

From Remark 3.1, we deduce

$$\ell_F(\varepsilon_{F,\mathfrak{m}}(\alpha))e_{\chi} \in \text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} C_{\mathfrak{m}})_{\chi} \right) (\mathcal{O}\ell_F(\mathcal{O}_{F \cap H_{\mathfrak{m}}, S_{\mathfrak{m}}}^{\times}))_{\chi}.$$

By Lemma 3.3, *ii*), and Remark 3.2, the norm map defines an isomorphism

$$(\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_{F,S_{\mathfrak{m}}}))_{\chi} \simeq (\mathcal{O} \otimes_{\mathbb{Z}} C_{\mathfrak{m}})_{\chi},$$

and the canonical inclusion $F \cap H_{\mathfrak{m}} \hookrightarrow F$ gives

$$(\mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}_{F \cap H_{\mathfrak{m}}, S_{\mathfrak{m}}}^{\times})_{\chi} = (\mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}_{F, S_{\mathfrak{m}}}^{\times})_{\chi}.$$

Then

$$\ell_F(\varepsilon_{F,\mathfrak{m}}(\alpha))e_{\chi} \in \text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_{F,S_{\mathfrak{m}}}))_{\chi} \right) (\mathcal{O}\ell_F(\mathcal{O}_{F,S_{\mathfrak{m}}}^{\times}))_{\chi}.$$

From Lemma 3.1, and Lemma 3.3, *iii*), we know that

$$\text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_{F,S_{\mathfrak{m}}}))_{\chi} \right) = \text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_{\chi} \right),$$

$$\text{and } (\mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}_{F,S_{\mathfrak{m}}}^{\times})_{\chi} = (\mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}_F^{\times})_{\chi}.$$

The proposition follows. □

4 Statement and proof of the theorems.

Theorem 4.1 *Let $\chi \in \widehat{G}$, $\chi \neq 1$. We have*

$$\text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_{\chi} \right) = \text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} (\mathcal{O}_F^{\times}/\mathcal{E}_F))_{\chi} \right).$$

Proof. We have

$$\begin{aligned} \prod_{\substack{\xi \in \widehat{G} \\ \xi \neq 1}} \text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_{\xi} \right) &= \text{Fit}_{\mathcal{O}} ((1 - e_1) \cdot \mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F)) \\ &= \mathcal{O} \# ((1 - e_1) \cdot \mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F)). \end{aligned} \quad (4.1)$$

In the same way, we have

$$\begin{aligned} \prod_{\substack{\xi \in \widehat{G} \\ \xi \neq 1}} \text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} (\mathcal{O}_F^{\times}/\mathcal{E}_F))_{\xi} \right) &= \mathcal{O} \text{Fit}_{\mathbb{Z}_{\langle g \rangle}} (\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} (\mathcal{O}_F^{\times}/\mathcal{E}_F)) \\ &= \mathcal{O} [\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \mathcal{O}_F^{\times} : \mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \mathcal{E}_F]. \end{aligned} \quad (4.2)$$

By (1.1), we know that

$$\# ((1 - e_1) \cdot \mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F)) = [\mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \mathcal{O}_F^{\times} : \mathbb{Z}_{\langle g \rangle} \otimes_{\mathbb{Z}} \mathcal{E}_F]. \quad (4.3)$$

Putting (4.1), (4.3) and (4.2) together, we obtain

$$\prod_{\substack{\xi \in \widehat{G} \\ \xi \neq 1}} \text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_{\xi} \right) = \prod_{\substack{\xi \in \widehat{G} \\ \xi \neq 1}} \text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} (\mathcal{O}_F^{\times}/\mathcal{E}_F))_{\xi} \right).$$

From this equality, it follows that it is sufficient to show the divisibility

$$\text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_{\xi} \right) \mid \text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} (\mathcal{O}_F^{\times}/\mathcal{E}_F))_{\xi} \right), \quad (4.4)$$

for all $\xi \in \widehat{G}$. Since $\mathcal{O}_F^{\times} \cap \text{Ker}(\ell_F) = \mu(F)$, we have $\mathcal{O}_F^{\times}/\mathcal{E}_F \simeq \ell_F(\mathcal{O}_F^{\times})/\ell_F(\mathcal{E}_F)$ and

$$(\mathcal{O} \otimes_{\mathbb{Z}} (\mathcal{O}_F^{\times}/\mathcal{E}_F))_{\xi} \simeq (\mathcal{O} \ell_F(\mathcal{O}_F^{\times}))_{\xi} / (\mathcal{O} \ell_F(\mathcal{E}_F))_{\xi}. \quad (4.5)$$

We set $\mathcal{F}_{\xi} := \text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_{\xi} \right)$ for convenience. From Proposition 3.1 we derive the tautological exact sequence

$$0 \longrightarrow \frac{\mathcal{F}_{\xi} (\mathcal{O} \ell_F(\mathcal{O}_F^{\times}))_{\xi}}{(\mathcal{O} \ell_F(\mathcal{E}_F))_{\xi}} \longrightarrow \frac{(\mathcal{O} \ell_F(\mathcal{O}_F^{\times}))_{\xi}}{(\mathcal{O} \ell_F(\mathcal{E}_F))_{\xi}} \longrightarrow \frac{(\mathcal{O} \ell_F(\mathcal{O}_F^{\times}))_{\xi}}{\mathcal{F}_{\xi} (\mathcal{O} \ell_F(\mathcal{O}_F^{\times}))_{\xi}} \longrightarrow 0. \quad (4.6)$$

Since \mathcal{O} is a Dedekind ring, we deduce from (4.6) and (4.5) that

$$\begin{aligned} \text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} (\mathcal{O}_F^{\times}/\mathcal{E}_F))_{\xi} \right) &= \text{Fit}_{\mathcal{O}} \left(\frac{\mathcal{F}_{\xi} (\mathcal{O} \ell_F(\mathcal{O}_F^{\times}))_{\xi}}{(\mathcal{O} \ell_F(\mathcal{E}_F))_{\xi}} \right) \text{Fit}_{\mathcal{O}} \left(\frac{(\mathcal{O} \ell_F(\mathcal{O}_F^{\times}))_{\xi}}{\mathcal{F}_{\xi} (\mathcal{O} \ell_F(\mathcal{O}_F^{\times}))_{\xi}} \right) \\ &= \text{Fit}_{\mathcal{O}} \left(\frac{\mathcal{F}_{\xi} (\mathcal{O} \ell_F(\mathcal{O}_F^{\times}))_{\xi}}{(\mathcal{O} \ell_F(\mathcal{E}_F))_{\xi}} \right) \mathcal{F}_{\xi}. \end{aligned} \quad (4.7)$$

□

Before we go further, we state here the definition and basic properties of index-modules, which we introduced in [9]. We refer the reader to [9] for the proofs.

Let K be a commutative field, $A \subseteq K$ be a Dedekind ring and V be a K -vector space. By an A -lattice of V , we mean a finitely generated A -submodule R of V , such that the K -vector subspace of V generated by R , denoted by KR , has dimension equal to the A -rank of R .

Definition 4.1 *Let $R \neq 0$ and S be A -lattices of V . We call A -index-module of the couple (R, S) the set*

$$[R : S]_A := \{\det(u); u \in \text{End}_K(V')/u(R) \subseteq S\},$$

where V' is the K -subspace of V generated by R and S , $V' = KR + KS$.

In fact, $[R : S]_A$ is an A -submodule of K , and we have the following properties, for any A -lattices R, S , and T of V , with $R \neq 0$.

i) $[R : R]_A = A$.

ii) If $KS \subseteq KR$, then $[R : S]_A$ is a finitely generated A -submodule of K . Moreover, its A -rank is 1 if $KR = KS$, and $[R : S]_A = 0$ if $KS \subsetneq KR$.

iii) Assume that $KR = KS$, and that \mathfrak{r} and \mathfrak{s} are two nonzero fractional ideals of A . Then $[\mathfrak{r}R : \mathfrak{s}S]_A = \mathfrak{s}^d \mathfrak{r}^{-d} [R : S]_A$, where d is the common A -rank of R and S .

iv) If $S \neq 0$, and $KT \subseteq KS \subseteq KR$, then $[R : T]_A = [R : S]_A [S : T]_A$.

v) If $S \subseteq R$, then $[R : S]_A = \text{Fit}_A(R/S)$.

In the sequel, we are concerned with the following situation. $A := \mathcal{O}$, $V := \mathbb{C}[G]$, and the \mathcal{O} -lattices are $(\mathcal{O}\ell_F(\mathcal{O}_F^\times))_\chi$, $(\mathcal{O}\ell_F(\mathcal{E}_F))_\chi$, $(\mathcal{O}\ell_F(\Omega))_\chi$, ..., where χ is a nontrivial complex character of G . They are all of \mathcal{O} -rank 1.

Lemma 4.1 *Let $\chi \in \widehat{G}$, $\chi \neq 1$. We have*

$$\left[(\mathcal{O}\ell(\Omega))_\chi : (\mathcal{O}\ell(\mathcal{E}_F))_\chi \right]_{\mathcal{O}} = w_\infty^{-1} \text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} \mu(F))_\chi \right),$$

where $\left[(\mathcal{O}\ell(\Omega))_\chi : (\mathcal{O}\ell(\mathcal{E}_F))_\chi \right]_{\mathcal{O}}$ and $w_\infty^{-1} \text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} \mu(F))_\chi \right)$ are viewed as \mathcal{O} -submodules of $\mathbb{Q}(\mu_g)$.

Proof. Let ζ be a primitive w_F -th root of unity in F , where $w_F := \#(\mu(F))$. We have an obvious exact sequence,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}\mathcal{J}_F e_\chi & \longrightarrow & \mathcal{O}e_\chi & \longrightarrow & (\mathcal{O} \otimes_{\mathbb{Z}} \mu(F))_\chi \longrightarrow 0, \\ & & & & \alpha e_\chi \longmapsto & & e_\chi(\alpha \otimes \zeta). \end{array}$$

Using the property v) of index-modules, we deduce

$$\text{Fit}_{\mathcal{O}} \left((\mathcal{O} \otimes_{\mathbb{Z}} \mu(F))_\chi \right) = [\mathcal{O}e_\chi : \mathcal{O}\mathcal{J}_F e_\chi]_{\mathcal{O}}. \quad (4.8)$$

Also, using (3.3), we have

$$\left[(\mathcal{O}\ell_F(\Omega))_\chi : (\mathcal{O}\ell_F(\mathcal{E}_F))_\chi \right]_\mathcal{O} = \left[(\mathcal{O}\ell_F(\Omega))_\chi : (\mathcal{O}\mathcal{J}_F w_\infty^{-1} \ell_F(\Omega))_\chi \right]_\mathcal{O}.$$

By (3.2), the property *iii*) of index-modules, and (4.8) for the last equality, we deduce

$$\begin{aligned} \left[(\mathcal{O}\ell_F(\Omega))_\chi : (\mathcal{O}\ell_F(\mathcal{E}_F))_\chi \right]_\mathcal{O} &= [\mathcal{O}w_\infty L_{\mathfrak{f}_\chi}(0, \bar{\chi}_{\text{pr}})e_\chi : \mathcal{O}\mathcal{J}_F L_{\mathfrak{f}_\chi}(0, \bar{\chi}_{\text{pr}})e_\chi]_\mathcal{O} \\ &= w_\infty^{-1} [\mathcal{O}e_\chi : \mathcal{O}\mathcal{J}_F e_\chi]_\mathcal{O} \\ &= w_\infty^{-1} \text{Fit}_\mathcal{O} \left((\mathcal{O} \otimes_{\mathbb{Z}} \mu(F))_\chi \right). \end{aligned}$$

□

The regulator $R(\mathcal{O}_F)$ of \mathcal{O}_F is known to be equal to $[\mathbb{Z}[G]_0 : \ell_F(\mathcal{O}_F^\times)]$, where $\mathbb{Z}[G]_0$ is the augmentation ideal of $\mathbb{Z}[G]$. Hence it is natural to take $R(\mathcal{O}_F)_\chi := [\mathcal{O}e_\chi : \ell_F(\mathcal{O}_F^\times) e_\chi]_\mathcal{O}$ as a definition for the « χ -part» of the regulator, for any nontrivial character $\chi \in \hat{G}$.

Theorem 4.2 *Let $\chi \in \hat{G}$, $\chi \neq 1$. Then we have*

$$R(\mathcal{O}_F)_\chi \text{Fit}_\mathcal{O} \left((\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_\chi \right) = \text{Fit}_\mathcal{O} \left((\mathcal{O} \otimes_{\mathbb{Z}} \mu(F))_\chi \right) L_{\mathfrak{f}_\chi}(0, \bar{\chi}_{\text{pr}}),$$

where $R(\mathcal{O}_F)_\chi \text{Fit}_\mathcal{O} \left((\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_\chi \right)$ and $\text{Fit}_\mathcal{O} \left((\mathcal{O} \otimes_{\mathbb{Z}} \mu(F))_\chi \right) L_{\mathfrak{f}_\chi}(0, \bar{\chi}_{\text{pr}})$ are viewed as \mathcal{O} -submodules of $\mathbb{Q}(\mu_g)$.

Proof. We keep the notation $\mathcal{F}_\chi := \text{Fit}_\mathcal{O} \left((\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_\chi \right)$. From Theorem 4.1, (4.5), and the property *v*) of index-modules, we have

$$\mathcal{F}_\chi = \text{Fit}_\mathcal{O} \left((\mathcal{O} \otimes_{\mathbb{Z}} (\mathcal{O}_F^\times / \mathcal{E}_F))_\chi \right) = \left[(\mathcal{O}\ell_F(\mathcal{O}_F^\times))_\chi : (\mathcal{O}\ell_F(\mathcal{E}_F))_\chi \right]_\mathcal{O}. \quad (4.9)$$

By the property *iv*) of index-modules, we deduce from (4.9) the decomposition

$$R(\mathcal{O}_F)_\chi \mathcal{F}_\chi = \left[\mathcal{O}e_\chi : (\mathcal{O}\ell_F(\Omega))_\chi \right]_\mathcal{O} \left[(\mathcal{O}\ell_F(\Omega))_\chi : (\mathcal{O}\ell_F(\mathcal{E}_F))_\chi \right]_\mathcal{O}. \quad (4.10)$$

By (3.2), and the property *iii*) of index-modules, we have

$$\left[\mathcal{O}e_\chi : (\mathcal{O}\ell_F(\Omega))_\chi \right]_\mathcal{O} = \mathcal{O}w_\infty L_{\mathfrak{f}_\chi}(0, \bar{\chi}_{\text{pr}}). \quad (4.11)$$

From (4.10), (4.11), and Lemma 4.1, we obtain

$$R(\mathcal{O}_F)_\chi \text{Fit}_\mathcal{O} \left((\mathcal{O} \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_\chi \right) = L_{\mathfrak{f}_\chi}(0, \bar{\chi}_{\text{pr}}) \text{Fit}_\mathcal{O} \left((\mathcal{O} \otimes_{\mathbb{Z}} \mu(F))_\chi \right).$$

□

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